The Y-Function[1]

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Abstract

Direct inference derives values for definite (single-case) probabilities from those of related indefinite (general) probabilities. But direct inference is less useful than might be supposed, because we often have too much information, with the result that we can make conflicting direct inferences, and hence they all undergo collective defeat, leaving us without any conclusion to draw about the value of the definite probabilities. This paper presents reason for believing that there is a function — the Y-function — that can be used to combine different indefinite probabilities to yield a single value for the definite probability. Thus we get a kind of "computational" direct inference.

1. Fundamentals of Direct Inference

This paper announces a new discovery which, I think, will be of fundamental importance in the application of probabilities to the real world. It concerns direct inference, so I will begin by sketching the theory of direct inference that I developed in my (1990).

There are two general approaches to probability theory. The most familiar takes "definite" or "single-case" probabilities to be basic. Definite probabilities attach to closed formulas or propositions. I write them using all caps: PROB(P) and PROB(P/Q). There are familiar difficulties for this approach, the simplest being that of making sense of the definite probabilities themselves. A serious practical difficulty is that when taking definite probabilities as basic, one must generally assume that we come to a problem equipped with a complete probability distribution. The latter is often computationally impossible for problems of realistic complexity. Like Kyburg (1974), I am convinced that the only way to construct a useful kind of definite probability is in terms of "indefinite" or "general" probabilities. The indefinite probability of *an A* being a *B* is not about any particular *A*, but rather about the property of being an *A*. In this respect, its logical form is the same as that of relative frequencies. "prob" is a variable-binding operator, binding the "x" in "prob(Bx/Ax)". I am convinced that the latter is the only approach that can provide the ultimate foundations for probability, in the sense of (1) providing a logical analysis of useful concepts of probability, and (2) explaining the epistemological foundations of probabilistic reasoning. The latter is required to make probabilities useful in science or AI.

According to this approach, statistical induction gives us knowledge of indefinite probabilities, and then direct inference gives us knowledge of definite probabilities. Reichenbach (1949) pioneered the theory of direct inference. The basic idea is that if we want to know the definite probability PROB(Fa), we look for the narrowest reference class (or reference property) G such that we know the indefinite probability prob(Fx/Gx) and we know Ga, and then we identify PROB(Fa) with prob(Fx/Gx). For example, actuarial reasoning aimed at setting insurance rates proceeds in this way. Kyburg (1974) was the first to attempt to provide firm logical foundations for direct inference, and my (1990) took that as its starting point and constructed a modified theory with a more epistemological orientation.[2] I will briefly sketch my own approach, and then discuss a general problem for theories of direct inference.

The appeal to indefinite probabilities and direct inference has seemed promising for avoiding the computational difficulties attendant on the need for a complete probability distribution. Instead of assuming that we come to a problem with an antecedently given complete distribution, one can

[1] This work was supported by NSF grant no. IIS-0412791.
[2] Competing theories of direct inference can be found in Levi (1980), Bacchus (1990), Halpern (1990).

more realistically assume that we come to the problem with some limited knowledge of indefinite probabilities, and then we infer definite probabilities from the latter as we need them. Unfortunately, as I will show in section two, it is premature to suppose that existing theories of direct inference actually solve this problem. The main point of this paper, however, is to exhibit a mathematical result that makes this problem solvable by appealing to a more powerful variety of direct inference — what I will call *computational direct inference*.

Kyburg used the term "probability" to refer only to definite probabilities, but I think it is better to distinguish between definite and indefinite probabilities. Kyburg identified indefinite probabilities with relative frequencies, but I think that is inadequate for a number of reasons (detailed in my 1990). The simplest is that we often make probability judgments that diverge from relative frequencies. For example, we can talk about a coin being a fair coin (and so the probability of a flip landing heads is 0.5) even when it is flipped only once and then destroyed. For understanding such indefinite probabilities, we need a notion of probability that talks about *possible* instances of properties as well as actual instances. My proposal in my (1990) was that we can identify the *nomic probability* prob(Fx/Gx) with the proportion of physically possible G's that are F's. A *physically possible* G is defined to be an ordered pair $\langle w,x\rangle$ such that w is a physically possible world (one compatible with all the physical laws) and x has the property G at w. $\mathfrak{G}$ is the set of all physically possible G's. We must assume the existence or a proportion function ρ(X,Y) for sets X and Y. I investigated the general theory of proportion in my (1990), although I no longer regard that treatment as entirely adequate, and am working on an improved version of that theory. For the moment, let us just assume we have the proportion function ρ. Then if it is physically possible for there to be G's, we define:

$$\text{prob}(Fx/Gx) = \rho(\mathfrak{F},\mathfrak{G}).$$

In my (1990) I proposed a generalization of this definition that handles the case of counterlegal probabilities, in which it is not physically possible for there to be G's, but I will ignore that sophistication here.

It will often be convenient to write proportions in the same logical form as probabilities, so where φ and θ are open formulas with free variable x, $\rho(\varphi/\theta) = \rho(\{x \mid \varphi \,\&\, \theta\},\{x \mid \theta\})$. Without going into details about the proportion function, I will make two classes of assumptions. Let #X be the cardinal of X. If Y is finite, I assume:

$$\rho(X,Y) = \frac{\#X \cap Y}{\#Y}.$$

However, for present purposes the proportion function is most useful in talking about proportions among infinite sets. The sets $\mathfrak{F}$ and $\mathfrak{G}$ will almost invariably be infinite, if for no other reason than that there are infinitely many physically possible worlds in which there are F's and G's. I assume that the standard "Boolean" principles that hold for finite relative frequencies also hold for proportions among infinite sets. In my (1990), I also defended various non-Boolean principles. In particular, I assume that principles that hold for finite sets in the limit (as their size goes to $\aleph_0$) hold for infinite sets. For instance, the following is a theorem of combinatorial mathematics:

Finite Principle of Agreement: For every $\varepsilon,\delta > 0$ there is a N such that if B is finite but $\#B > N$ then

$$\rho\Big(\rho(A,X) \underset{\delta}{\approx} \rho(A,B) / X \subseteq B\Big) \geq 1-\varepsilon.$$

In other words, proportions among subsets of B tend to agree with proportions in B itself, and both the strength of the tendency and the extent of the agreement increase as the size of B increases. I take this to support a corresponding principle of agreement for infinite sets:

Strong Principle of Agreement: For every $\delta > 0$, if B is infinite then

$$\rho\Big(\rho(A,X) \underset{\delta}{\approx} \rho(A,B) / X \subseteq B\Big) = 1.$$

In my (1990) I showed that this principle is derivable from the general theory of proportions adumbrated there, although as I remarked above, I no longer regard that theory as entirely adequate. Nomic probabilities are proportions among physically possible objects, so the Strong Principle of Agreement implies:

Principle of Agreement for Probabilities: For every $\delta > 0$,

$$\text{prob}\Big(\text{prob}(Ax / Bx) \underset{\delta}{\approx} \text{prob}(Ax / Gx) \,/\, B \preccurlyeq G\Big) = 1.$$

I take the Principle of Agreement for Probabilities to underlie direct inference, as I will now explain.

The theory of direct inference is an epistemological theory. It is about how to make certain kinds of inferences. In my (1990) I showed that the entire epistemological theory of nomic probability can be derived from a single epistemological principle coupled with a mathematical theory that amounts to a calculus of nomic probabilities. The latter is richer than the standard probability calculus, because nomic probabilities have more structure than definite probabilities. The relationship between indefinite probabilities and definite probabilities is roughly analogous to the difference between the predicate calculus and the propositional calculus. I won't pursue the details of the calculus here, but see my (1990). The single epistemological principle that underlies probabilitistic reasoning is the *statistical syllogism*. The basic form of the statistical syllogism that I will employ here is the modified version defended in my (1995):

(A3) If G is projectible with respect to both K and U and $r > 0.5$, then $\ulcorner Kc$ & prob$(Gx/Kx\&Ux) \geq r\urcorner$ is a defeasible reason for $\ulcorner(Uc \rightarrow Gc)\urcorner$.[3]

I assume the theory of defeasible reasoning adumbrated in my (1995).

When U is tautologous, (A3) implies the simpler:

(A1) If G is projectible with respect to K and $r > 0.5$, then $\ulcorner Kc$ & prob$(Gx/Kx) \geq r\urcorner$ is a defeasible reason for $\ulcorner Gc\urcorner$.

The simplest kind of defeater is a subproperty defeater:

(D) If H is projectible with respect to both K and U, $\ulcorner Hc$ & prob$(Gx/Kx\&Ux\&Hx) \neq$ prob$(Gx/Kx\&Ux)$ & prob$(Ux/Kx\&Hx) \geq$ prob$(U/K)\urcorner$ is an undercutting defeater for (A1) and (A3).[4]

In (A1), U is tautologous, so in that case (D) can be simplified:

(D1) If H is projectible with respect to K, $\ulcorner Hc$ & prob$(Gx/Kx\&Hx) \neq$ prob$(Gx/Kx)\urcorner$ is an undercutting defeater for (A1).

More defeaters are required to make inferences in accordance with the statistical syllogism work properly. Several are discussed in my my (1990), but for present purposes we need not pursue this.

The statistical syllogism, coupled with the principle of agreement, gives us a defeasible reason for expecting that if prob$(Fx/Gx) = r$ then for every $\delta > 0$, prob$(Fx/Gx\&Hx) \approx_{\delta} r$, and the latter entails that prob$(Fx/Gx\&Hx) = r$. Thus we get a general principle of direct inference, which I call *nonclassical direct inference*:

Nonclassical Direct Inference:

If F is projectible with respect to G, $\ulcorner$prob$(Fx / Gx) = r\urcorner$ is a defeasible reason for $\ulcorner$prob$(Fx / Gx \,\&\, Hx) = r\urcorner$.

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[3] The projectibility constraint is the familiar constraint required for inductive reasoning. This is discussed at length in my (1990). Kyburg (1974) also noted the need for some such constraint.

[4] There are two kinds of defeaters. Rebutting defeaters attack the conclusion of an inference, and rebutting defeaters attack the inference itself without attacking the conclusion (Pollock 1995). Here I assume some form of the OSCAR theory of defeasible reasoning.

This is a kind of principle of insufficient reason. It differs from classical direct inference in that it is an inference from indefinite probabilities to indefinite probabilities rather than from indefinite probabilities to definite probabilities.

Defeaters for nonclassical direct inference follow from the defeaters for the statistical syllogism. In particular, subproperty defeaters for the statistical syllogism generate subproperty defeaters for nonclassical direct inference. Let us define:

$F \preccurlyeq G$ iff it is physically necessary that $(\forall x)(Fx \rightarrow Gx)$.

$F \prec G$ iff $F \preccurlyeq G$ but $\sim(G \preccurlyeq F)$.

Then subproperty defeat for (A1) generates subproperty defeat for nonclassical direct inference:

<u>**Subproperty Defeat**</u>:

⌜ $G \prec J \prec (G \,\&\, H)$ and $\text{prob}(Fx/Jx) \neq r$ ⌝ is an undercutting defeater for nonclassical direct inference.

Direct inference is normally understood as being a form of inference from indefinite probabilities to definite probabilities rather than from indefinite probabilities to other indefinite probabilities. However, I showed in my (1990) that these inferences are derivable from nonclassical direct inference if we identify definite probabilities with a special class of indefinite probabilities. Let $\boldsymbol{K}$ be the conjunction of the agent's justified beliefs. Then we define:

PROB$(P/Q) = r$ iff for some n, there are n-ary properties R and S and terms $a_1,\ldots,a_n$ such that
(1) it is physically necessary that $(P \leftrightarrow Ra_1\ldots a_n)$ and $(Q \leftrightarrow Sa_1\ldots a_n)$, and
(2) $\text{prob}(Rx_1\ldots x_n / Sx_1\ldots x_n \,\&\, x_1 = a_1 \,\&\, \ldots \,\&\, x_n = a_n \,\&\, \boldsymbol{K}) = r$.

It can be shown (my 1990) that this yields a unique value for PROB(P/Q). This is a kind of "mixed physical/epistemic probability", because it combines background knowledge in the form of $\boldsymbol{K}$ with indefinite probabilities.

Given this definition, we can derive the *principle of classical direct inference:*

⌜ $\text{prob}(Rx_1\ldots x_n/Tx_1\ldots x_n) = r$, it is physically necessary that $(P \leftrightarrow Ra_1\ldots a_n)$ and that $(Q \leftrightarrow Sa_1\ldots a_n)$, and $Sx_1\ldots x_n \prec Tx_1\ldots x_n$ ⌝ is a defeasible reason for ⌜ PROB$(P/Q) = r$ ⌝.

Similarly, we get subproperty defeaters:

⌜ $S \prec U \prec T$ and $\text{prob}(Rx/Ux) \neq r$ ⌝ is an undercutting defeater for classical direct inference.

All of this is only a brief sketch of the theory of direct inference developed in my (1990), but it will suffice for present purposes. In the next section I will discuss a problem for this theory. The main part of this paper will be aimed at establishing the existence of a mathematical function that provides a solution to the problem and extends the theory of direct inference in a way that, perhaps for the first time, makes it truly useful.

2. A Problem for Direct Inference

The preceding provides a foundation for a more or less standard theory of direct inference. Perhaps its main novelty is that it is formulated in terms of a background theory of defeasible reasoning. However, this and all similar theories suffer from a fundamental difficulty that greatly diminishes their practical usefulness. If we have some complex conjunction $G_1x \,\&\ldots\&\, G_nx$ of properties and we want to know the value of $\text{prob}(Fx/G_1x \,\&\ldots\&\, G_nx)$, if we know that $\text{prob}(Fx/G_1x) = r$ *and we don't know anything else of relevance*, we can infer defeasibly that $\text{prob}(Fx/G_1x \,\&\ldots\&\, G_nx) = r$. Similarly, if we know that an object a has the properties $G_1,\ldots,G_n$ and we know that $\text{prob}(Fx/G_1x) = r$ *and we don't know anything else of relevance*, we can infer defeasibly that PROB$(Fa) = r$. The difficulty is that we usually know more. We typically know the value of $\text{prob}(Fx/G_ix)$ for some $i \neq 1$. If $\text{prob}(Fx/G_ix) = s \neq r$, we have defeasible reasons for both ⌜ $\text{prob}(Fx/G_1x \,\&\ldots\&\, G_nx) = r$ ⌝ and ⌜ $\text{prob}(Fx/G_1x \,\&\ldots\&\, G_nx) = s$ ⌝, and also for both ⌜ PROB$(Fa) =$

r⌝ and ⌜PROB(Fa) = s⌝, but as these conclusions are incompatible they all undergo collective defeat. Thus the standard theory of direct inference leaves us without a conclusion to draw. The upshot is that the promise of direct inference to solve the computational problem of dealing with definite probabilities without having to have a complete probability distribution was premature. Direct inference will rarely give us the probabilities we need.

Direct inference would be vastly more useful in real application if there were a function Y(r,s) such that, in a case like the above, we could defeasibly expect that prob(Fx/G_1x &…& G_nx) = Y(r,s), and hence (by nonclassical direct inference) that PROB(Fa) = Y(r,s). I call this *computational direct inference*, because it computes a new value for PROB(Fa) rather than simply taking a value from a known indefinite probability. I call the function used in such a computation "the Y-function" because its behavior would be as diagrammed in figure 1. The general presumption has been that there is no such function, but this paper presents empirical reasons for thinking that the Y-function exists. I will present these reasons and then indicate how the existence of the Y-function can give rise to a theory of computational direct inference.

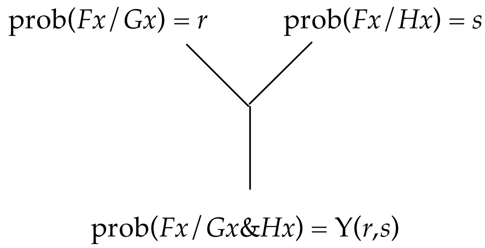

Figure 1. The Y-function

3. Discovering the Y-function

It is generally assumed that no such function as the Y-function exists. Certainly, there is no function Y(r,s) such that we can conclude deductively that if prob(Fx/Gx) = r and prob(Fx/Hx) = s then prob(Fx/Gx&Hx) = Y(r,s). For any r and s that are neither 0 nor 1, prob(Fx/Gx&Hx) can take any value between 0 and 1. However, that is equally true for nonclassical direct inference. That is, if prob(Fx/Gx) = r we cannot conclude deductively that prob(Fx/Gx&Hx) = r. Nevertheless, that will tend to be the case, and we can defeasibly expect it to be the case. Might something similar be true of the Y-function? That is, could there be a function Y(r,s) such that we can defeasibly expect prob(Fx/Gx&Hx) to be Y(r,s)?

I had always sided with the majority in supposing that there is no such function, but it occurred to me that perhaps this was premature. I did not (and do not yet) see how to prove the existence of the Y-function, but we might investigate its existence empirically. Let me explain.

On analogy with the principle of agreement, suppose there were a function Y(r,s) satisfying the following principle:

Y-Principle: For every $\varepsilon,\delta > 0$ there is a N such that if U is finite and $\#U > N$ then

$$\rho\left(f_3 \underset{\delta}{\approx} Y(f_1,f_2) \,/\, f_2 = \rho(A,B)\ \&\ f_3 = \rho(A,C)\ \&\ f_1 = \rho(A,B\cap C)\ \&\ A,B,C \subseteq U\right) \geq 1-\varepsilon.$$

Then in accordance with my general assumptions about the proportion function, it could be assumed that the following infinite generalization holds:

Strong Y-Principle: For every $\delta > 0$, if U is infinite then

$$\rho\left(f_3 \underset{\delta}{\approx} Y(f_1,f_2) \,/\, f_2 = \rho(A,B)\ \&\ f_3 = \rho(A,C)\ \&\ f_1 = \rho(A,B\cap C)\ \&\ A,B,C \subseteq U\right) = 1.$$

Nomic probabilities are proportions among physically possible objects. If there are infinitely many physically possible G's (a condition satisfied by pretty much any property you can come up with, because $\mathscr{G}$ can be infinite just by there being infinitely many *w's* entering into the ordered pairs ⟨*w*,*x*⟩), we then have the following:

Y-Principle for Probabilities: For every $\delta > 0$, if *U* is infinite then (where *A, B* and *C* are variables):

$$\text{prob}\left(f_3 \underset{\delta}{\approx} Y(f_1, f_2) \;/\; f_2 = \text{prob}(Ax / Bx) \;\&\; f_3 = \text{prob}(Ax / Cx) \;\&\; f_1 = \text{prob}(Ax / Bx \;\&\; Cx) \;\&\; A, B, C \preccurlyeq G \right) = 1.$$

The Y-Principle combined with (A1) will yield the following principle of direct inference in the same way the Principle of Agreement combined with (A1) yields the standard principle of nonclassical direct inference:

Computational Direct Inference:

If *F* is projectible with respect to *G* and *H*, ⌜prob$(Fx / Gx) = r$ & prob$(Fx / Hx) = s$⌝ is a defeasible reason for ⌜prob$(Fx / Gx \;\&\; Hx) = Y(r, s)$⌝.

Thus to get a principle of computational direct inference, it suffices to have a function Y(*r*,*s*) that satisfies the Y-principle for finite sets. Is there such a function? Not seeing how to prove that there is (or isn't) such a function, I decided to test this empirically. Given a set *U*, we could in principle survey all triples A,*B*,C of subsets of *U*, compute ρ(A,C), ρ(A,C), and ρ(A,*B*∩C) and see how they are related. If the Y-Principle seems to hold for larger and larger *U*, that gives us an inductive reason for thinking that the Y-Principle is true. The trouble is, you can only survey all triples of *U* for very small *U*. For instance, if #*U* = 100, the number of triples A,*B*,C of subsets of *U* is 2^{300}, which is approximately 10^{90}. This is twelve orders of magnitude greater than a recent estimate of the total number of elementary particles in the universe (10^{78}). Clearly, we cannot survey all these triples.

Although for even rather small *U's*, we cannot survey all of the triples of subsets, we can instead use Monte Carlo techniques. That is, we can sample the triples randomly and see what we get. Let Num(*k*) be the set of integers {1,…,*k*}. I wrote a program in LISP that randomly selects triples of nonempty subsets of Num(*k*) for any *k* we supply, and then computes and compares the values of ρ(A,C), ρ(A,C), and ρ(A,*B*∩C). To my surprise, this produced the plot of a very well-behaved function. The plot of 10,000 triples for *k* = 10,000 is given in figure 2. This plots the average value of ρ(A,*B*∩C) as a function of ρ(A,C) and ρ(A,C).

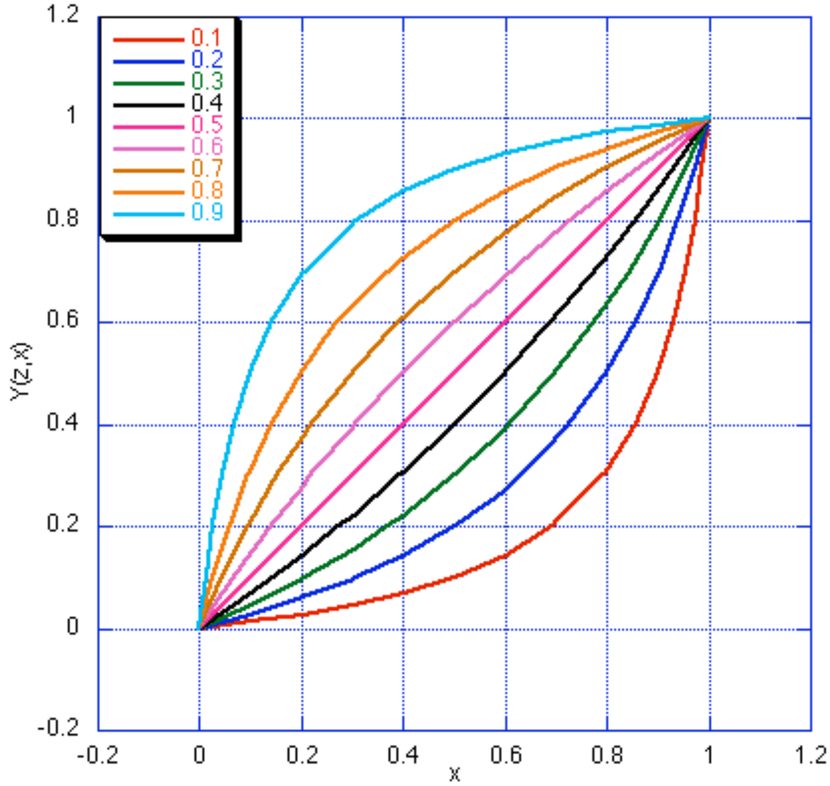

Figure 2. The Y-function, holding *z* constant

Although the average values are described by a well-behaved function, which I will designate *the Y-function*, this does not show that the value of ρ(A,*B*∩C) tends to agree with Y(ρ(A,*B*), ρ(A,C)). All it shows is that the *average value* conforms to the Y-function. However, this is also a matter we can investigate empirically. We can construct our sampling function so that it not only computes the average value of ρ(A,*B*∩C) for each choice of values for ρ(A,*B*) and ρ(A,C), but it also keeps track of how many of the values agree with Y(ρ(A,*B*), ρ(A,C)) to any specified degree of approximation. The results are quite striking. They are diagrammed in figures 3, 4, and 5. Inspection reveals that the envelopes containing 95% of the sampled triples get narrow rapidly.

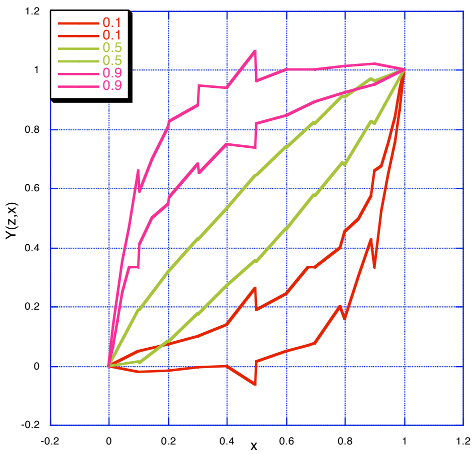

Figure 3. Upper and lower bounds for Y(z,x) (for three values of z) within which 95% of the sampled triples fall for $k = 100$.

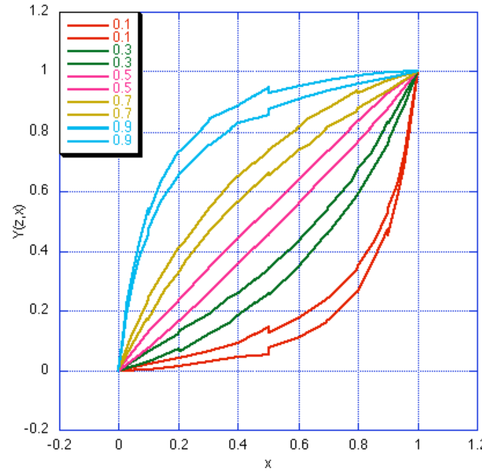

Figure 4. Upper and lower bounds for Y(z,x) (for five values of z) within which 95% of the sampled triples fall for $k = 1000$.

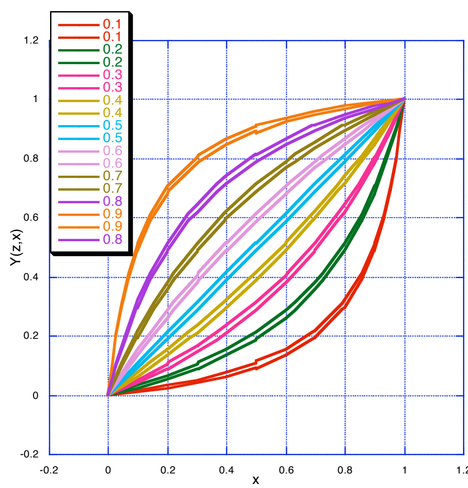

Figure 5. Upper and lower bounds for Y(z,x) within which 95% of the sampled triples fall for $k = 10{,}000$.

These results give strong inductive support for the existence of a Y-function satisfying the Y-Principle. It remains a mystery what function this is. I have not been able to find any familiar function the fits the curve of figure 2. Because this function is generated by statistical processes related to normal distributions, it may have no analytic characterization.

4. Algebraic Properties of the Y-function

The most important task facing us is to identify the Y-function and prove the Y-Principle. At this point, I can do neither. However, it may be useful to investigate the algebraic properties of the Y-function that can be read off of figure 2. These may ultimately be helpful in identifying the Y-function.

Clearly, from its definition, the Y-function must be associative:

(1) $\quad Y(z,x) = Y(x,z)$.

This property is also exhibited by the plot, as illustrated in figure 6.

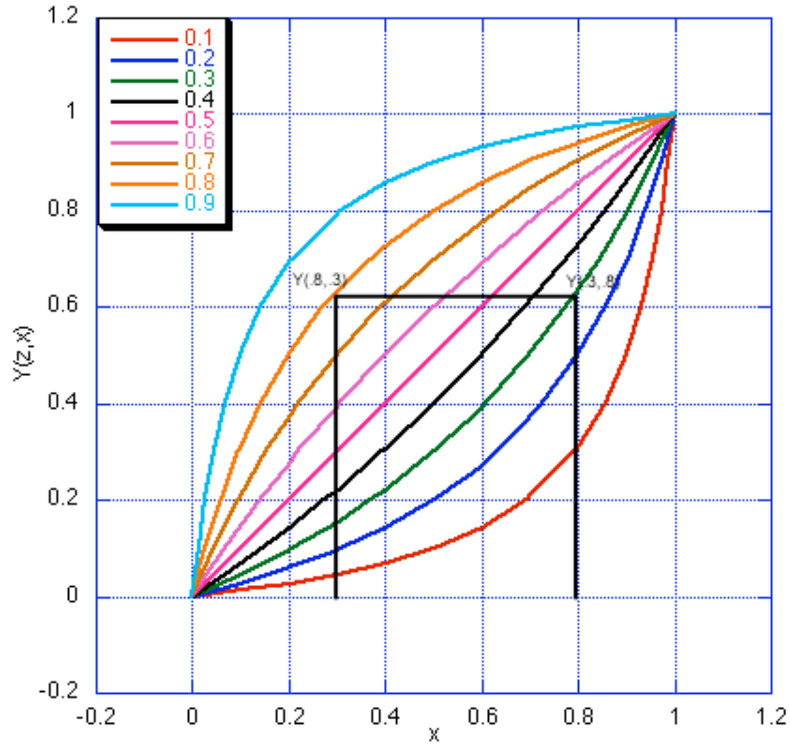

Figure 6. Associativity.

Principle (2) tells us that the value .5 plays a special role in the Y-function:

(2) Y(.5+x,.5-x) = .5.

Equivalently:

(3) Y(z,1-z) = .5

Principles (2) and (3) are illustrated by figure 7.

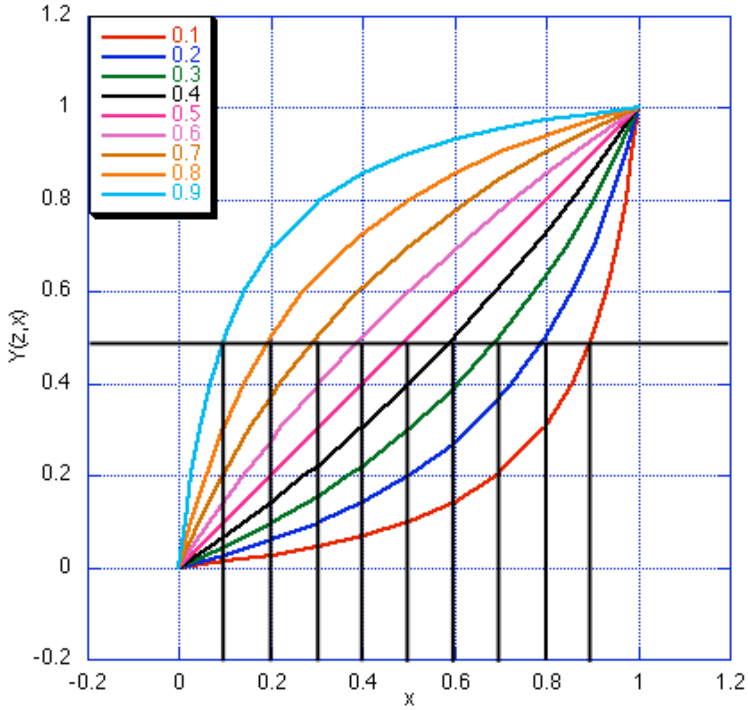

Figure 7. Y(*z*,1-*z*) = .5.

Principle (4) expresses another respect in which .5 plays a privileged role.

(4) Y(.5,*x*) = *x*

Principle (5) is diagrammed in figure 8.

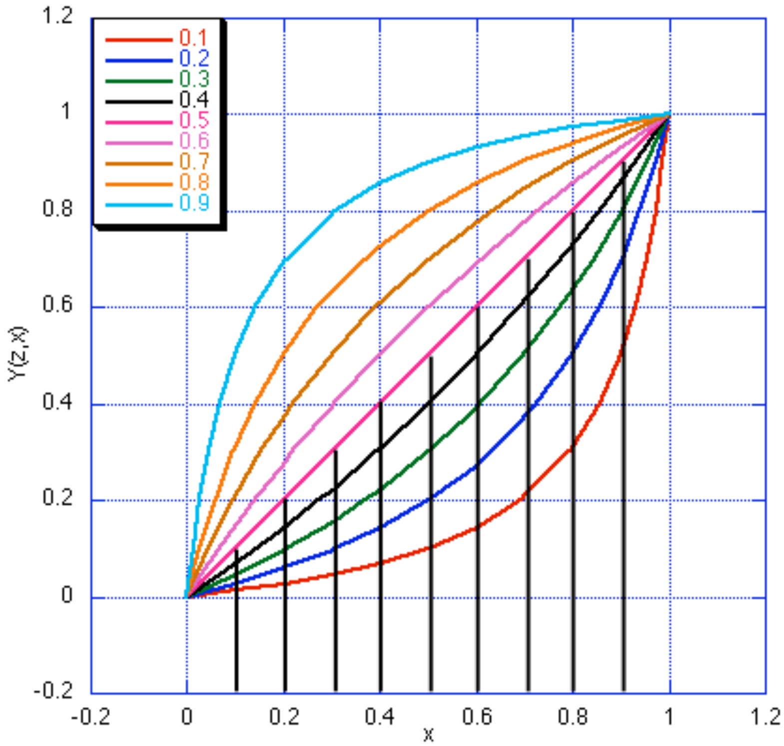

Figure 8. Y(.5,*x*) = *x*.

The curve exhibits symmetry in three directions — around both diagonals and around the center point. Symmetry around the center point is expressed by principle (6), which is diagrammed in figure 9:

(5) $Y(z,x) = 1–Y(1–z,1–x)$

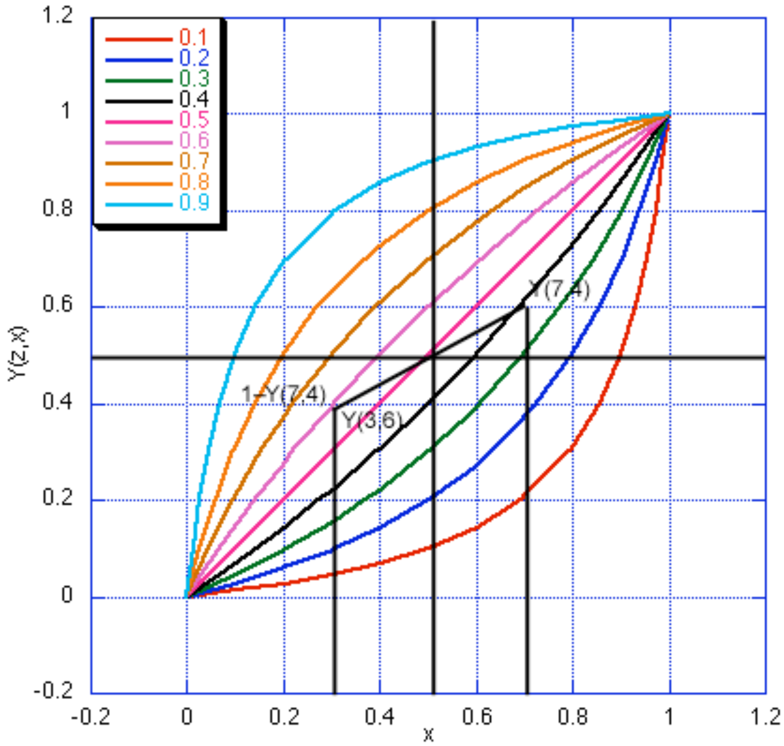

Figure 9. Y(*z*,*x*) = 1–Y(1–*z*,1–*x*)

Symmetry around the right-leaning diagonal is expressed by principle (6), which is diagrammed by figure 10:

(6) Y(1–*z*,Y(*z*,*x*)) = *x*

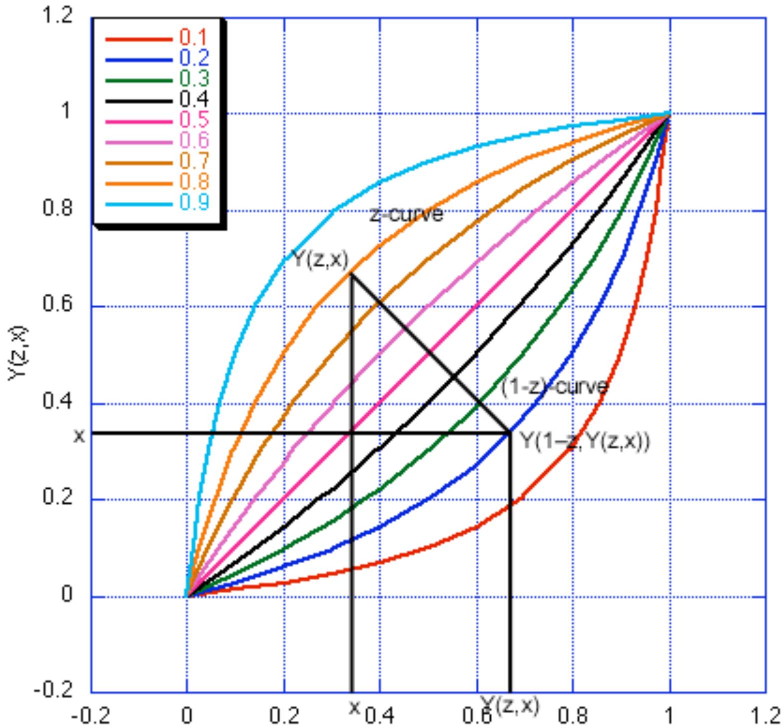

Figure 10. Y(1–z,Y(z,x)) = x

Principles (5) and (6) entail principle (7), which expresses symmetry around the left-leaning diagonal:

(7) Y(z,1–Y(z,x)) = 1–x.

Proof: Y(z,1–Y(z,x)) = 1–(1–Y(1–(1–z),1–Y(z,x)) = 1–Y(1–z,Y(z,x)) = 1–x.

Principle (7), and its derivation, is diagrammed in figure 11.

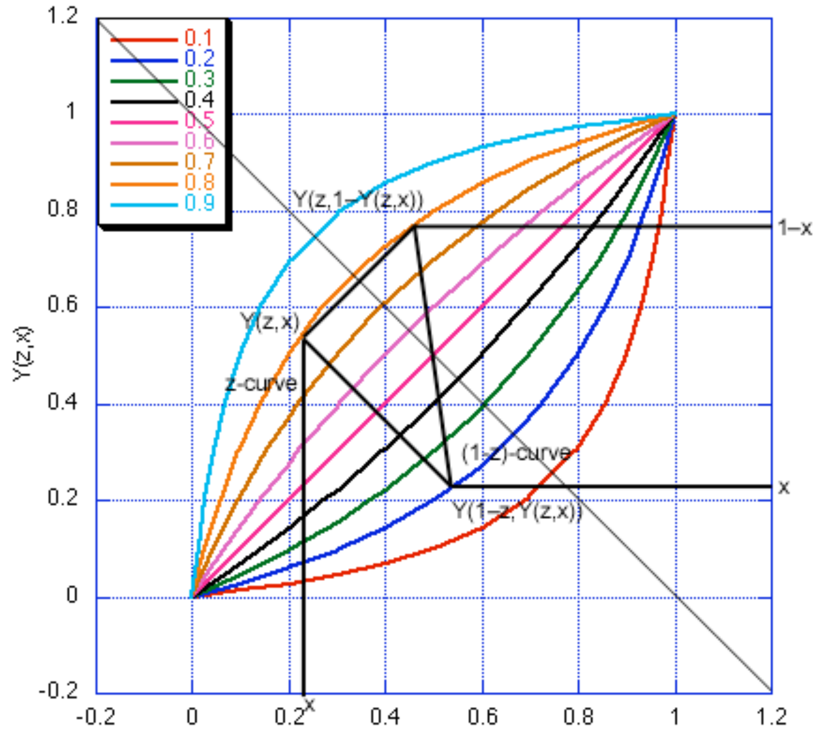

Figure 11. Y(z,1–Y(z,x)) = 1–x.

A principle that will prove very important is the following:

(8) Y(z,Y(x,w)) = Y(w,Y(z,x))

Principle (8) is illustrated by figure 12, although it does not reflect any obvious geometric properties of the curve.

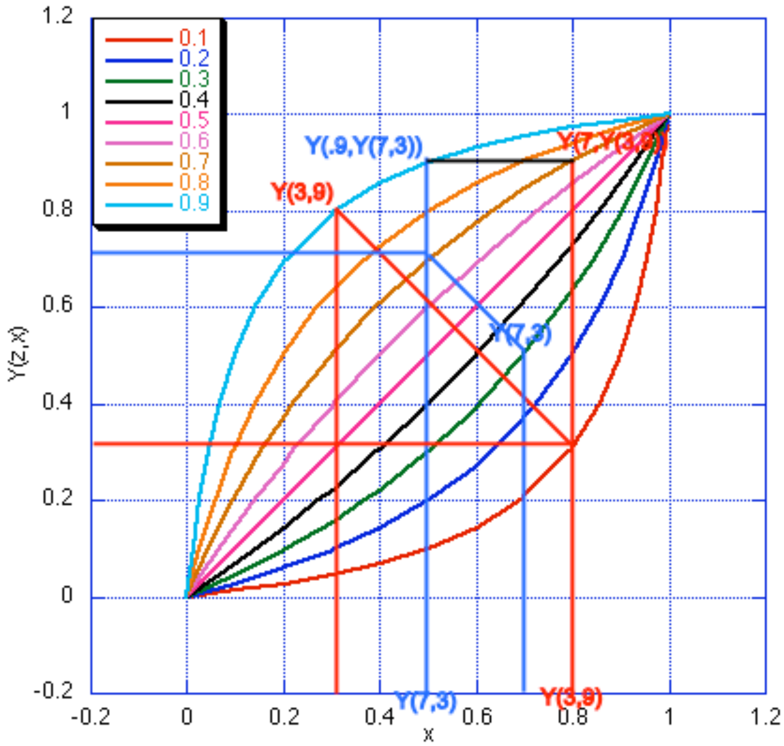

Figure 12. Y(*z*,Y(*x*,w)) = Y(w,Y(*z*,*x*)).

The reason principle (8) is important is that, together with associativity, it entails the commutativity of the Y-function:

(9) Y(*z*,Y(*x*,w)) = Y(Y(*z*,*x*),w).

Note that the Y-Principle also entails (9). So if the Y-Principle is correct, commutativity follows.

These algebraic properties describe a very well-behaved function, but it is still not clear what function it is.

5. Computational Direct Inference

The existence of the Y-function is a fundamental discovery that makes direct inference useful in ways it was never previously useful. The Y-principle tells us how to combine different probabilities in direct inference and still arrive at a univocal value. The form of the Y-function is diagrammed as in figure 13.

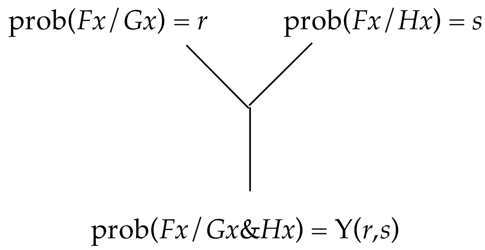

Figure 13. The Y-function

Thus, for example, if we know that the probability of Jones' dying if we shoot him is .6 and the probability of his dying if we poison him is .7, we can infer defeasibly (by A1) that the probability of his dying if we do both is Y(.6,.7) = .77778. This explains how built-in redundancy can increase the probability of a plan achieving its goal.

As observed in section three, the Y-Principle combined with (A1) yields the following principle of direct inference in the same way the Principle of Agreement combined with (A1) yields the standard principle of nonclassical direct inference:

Computational Direct Inference:

If F is projectible with respect to G and H, $\ulcorner \text{prob}(Fx / Gx) = r \;\&\; \text{prob}(Fx / Hx) = s \urcorner$ is a defeasible reason for $\ulcorner \text{prob}(Fx / Gx \;\&\; Hx) = Y(r,s) \urcorner$.

If we know that $\text{prob}(Fx/Gx) = r$ and $\text{prob}(Fx/Hx) = s$, we can also use nonclassical direct inference to infer defeasibly that $\text{prob}(Fx/Gx\&Hx) = r$ and $\text{prob}(Fx/Gx\&Hx) = s$, and this conflicts with the conclusion that $\text{prob}(Fx/Gx\&Hx) = \text{Y}(r,s)$. However, these conflicting conclusions are obtained by applying (A1) to weaker reference properties, and so they are defeated by subproperty defeat. In general:

Computational Defeat for Classical Direct Inference:

If F is projectible with respect to H, $\ulcorner \text{prob}(Fx / Hx) = s \urcorner$ is an undercutting defeater for nonclassical direct inference.

Computational direct inference is subject to subproperty defeat in the same way nonclassical direct inference is, and for the same reason:

$\ulcorner\, G \prec J \prec (G \,\&\, H)$ and $\text{prob}(Fx/Jx) \neq r \,\urcorner$ is an undercutting defeater for computational direct inference.

$\ulcorner\, H \prec J \prec (G \,\&\, H)$ and $\text{prob}(Fx/Jx) \neq s \,\urcorner$ is an undercutting defeater for computational direct inference.

For its use in computing probabilities, it is very important that the Y-function is commutative. If we know that $\text{prob}(Fx/Ax) = .6$, $\text{prob}(Fx/Bx) = .7$, and $\text{prob}(Fx/Cx) = .75$, we can combine them in any order to infer defeasibly that $\text{prob}(Fx/Ax\&Bx\&Cx) = \text{Y}(.6,\text{Y}(.7,.75)) = \text{Y}(\text{Y}(.6,.7),.75) = .913043$. This makes it convenient to extend the Y-function recursively so that it can be applied to an arbitrary number of arguments (greater than or equal to 1):

If $n > 2$, $\text{Y}(r_1,\ldots,r_n) = \text{Y}(r_1,\text{Y}(r_2,\ldots,r_n))$.

Then we can strengthen computational direct inference as follows:

Computational Direct Inference:

If F is projectible with respect to $G_1,\ldots,G_n$, $\ulcorner \text{prob}(Fx / G_1x) = r_1 \;\&\; \text{prob}(Fx / G_nx) = r_n \urcorner$ is a defeasible reason for $\ulcorner \text{prob}(Fx / G_1x \;\&\; \ldots \;\&\; G_nx) = Y(r_1,\ldots,r_n) \urcorner$.

Defeaters are derivable from those for "binary" computational direct inference.

5. Conclusions

The use of indefinite probabilities and direct inference seems initially to provide a computationally feasible alternative to the unrealistic requirement that we come to problems equipped with a complete distribution of definite probabilities. However, the initial promise fades with the realization that we normally have too much information, with the result that we can make conflicting inferences by nonclassical direct inference, and they collectively defeat one another. This difficulty is resolved by the discovery of the Y-function and the principle of computational direct inference, which allows us to make use of all our information to compute a single probability. As yet, we have only inductive reasons for believing that the Y-function exists, and we do not have an analytic characterization of it. Hopefully, future research will fill these lacunae.

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